

HOMOTOPY CLASSIFICATION OF CONTACT FOLIATIONS ON AN OPEN CONTACT MANIFOLD

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ABSTRACT. We give a complete classification of foliations on open contact manifolds whose leaves are contact submanifolds of the ambient manifold. The results are analogues of Haefliger's classification of foliations on open manifold.

1. INTRODUCTION

Foliations are important geometric structures on manifolds which define decompositions of the manifold into injectively immersed submanifolds, called leaves. In the present article we study foliations in the presence of a geometric structure on the manifold so that the leaves inherit similar structures from the ambient manifold. The simplest type of regular foliations on a manifold M are obtained from submersions on it. In this case the level sets of the submersions define the leaves of regular foliations on M . More generally if a map $f : M \rightarrow N$ is transversal to a foliation \mathcal{F}_N on N then the inverse image of the \mathcal{F}_N under f is a foliation on M . Haefliger proved that any foliation on M can actually be defined as the inverse image of a foliation by a map into a foliated manifold which is transversal to the foliation.

There is a notion of Γ -structure on topological spaces for any topological groupoid Γ (see Section 7). Following Milnor's topological join construction to define classifying space of principal G -bundles, one can construct a topological space $B\Gamma$ which classifies Γ structures ([11], [13]). In particular, when $\Gamma = \Gamma_q$ is the groupoid of germs of local diffeomorphisms of \mathbb{R}^q , the derivative map $d : \Gamma_q \rightarrow GL_q(\mathbb{R})$ induces a continuous map $Bd : B\Gamma_q \rightarrow BGL_q(\mathbb{R})$, where $BGL_q(\mathbb{R})$ is the classifying space of rank q real vector bundles. Haefliger cocycles defining a foliation \mathcal{F} on M naturally give rise to a Γ_q -structure on M . If \tilde{f} is a classifying map of the associated Γ_q structure then $Bd \circ \tilde{f}$ classifies the normal bundle of the foliation \mathcal{F} . Conversely, if a map $f : M \rightarrow BGL(q)$ classifying the normal bundle of a codimension q distribution D on M lifts to $B\Gamma_q$, then the distribution D is homotopic to one which is integrable, provided M is open. Haefliger showed that there is a vector bundle $\pi : E\Gamma_q \rightarrow B\Gamma_q$ such that the integrable homotopy classes of foliations on M can be classified by homotopy classes of bundle epimorphisms $(F, f) : TM \rightarrow E\Gamma_q$. The latter space is in one to one correspondence with the space of homotopy classes of lifts of a classifying map of the tangent bundle in $B\Gamma_q \times BGL_{n-q}(\mathbb{R})$.

In [3], we had given a homotopy classification of regular contact foliations on a manifold. In this article we study foliations on open contact manifolds for which the leaves are contact submanifolds of the ambient manifold. Let M^{2m+1} be a manifold with a contact form α . Recall that a 1-form α on an odd dimensional manifold M^{2n+1} is a contact form if $\alpha \wedge (d\alpha)^n$ is nowhere vanishing. It follows that the restriction of the 2-form $d\alpha$ to the subbundle $\xi = \ker \alpha$ defines a

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symplectic structure on the bundle. A foliation \mathcal{F} on (M, α) will be called a *contact foliation on M subordinate to α* (or simply a *contact foliation*) if the leaves of \mathcal{F} are contact submanifolds of M . The tangent distribution of a contact foliation on (M, α) is transverse to the contact subbundle $\xi = \ker \alpha$. Moreover, the intersection $T\mathcal{F} \cap \xi$ defines a symplectic subbundle of ξ with respect to the symplectic form $d'\alpha = d\alpha|_{\xi}$. Suppose that N^{2n} is any manifold with a regular foliation \mathcal{F}_N , where $n < m$. Let $Tr_{\alpha}(M, \mathcal{F}_N)$ be the space of all maps $f : M \rightarrow N$ which are transversal to \mathcal{F}_N and such that $f^{-1}\mathcal{F}_N$, the inverse image of the foliation \mathcal{F}_N , is a contact foliation on M . Let $\mathcal{E}_{\alpha}(TM, \nu\mathcal{F}_N)$ be the space of all vector bundle morphisms $F : TM \rightarrow TN$ such that

- (1) $q \circ F : TM \rightarrow \nu(\mathcal{F}_N)$ is an epimorphism, where $\nu(\mathcal{F}_N)$ is the normal bundle to the foliation \mathcal{F}_N ,
- (2) $\ker(q \circ F)$ is transverse to the contact distribution $\ker \alpha$ and
- (3) $\ker(q \circ F) \cap \ker \alpha$ is a symplectic subbundle of $(\ker \alpha, d'\alpha)$.

With C^{∞} -compact open topology on $Tr_{\alpha}(M, \mathcal{F}_N)$ and C^0 -compact open topology on $\mathcal{E}_{\alpha}(TM, \nu\mathcal{F}_N)$ we obtain the following result.

Theorem 1.1. *Let (M^{2m+1}, α) be an open contact manifold and N^{2n} , $m > n$, a foliated manifold with a regular foliation \mathcal{F}_N . Then the derivative map $d : Tr_{\alpha}(M, \mathcal{F}_N) \rightarrow \mathcal{E}_{\alpha}(TM, \nu\mathcal{F}_N)$ is a weak homotopy equivalence.*

Theorem 1.1 leads to the following classification of contact foliations on open contact manifolds.

Theorem 1.2. *Let (M, α) be an open contact manifold. The integral homotopy classes of codimension $2q$ contact foliations \mathcal{F} on M subordinate to α are in one-one correspondence with the homotopy classes of bundle epimorphisms $(F, f) : TM \rightarrow E\Gamma_{2q}$ for which $\ker F$ is transversal to $\ker \alpha$ and $\ker F \cap \ker \alpha$ is a symplectic subbundle of $\ker \alpha$.*

Theorem 1.1 may be viewed as a contact version of the Gromov-Phillips Theorem [17]. A Symplectic analogue of this result was proved in [2] but the problem of classification of symplectic foliations on symplectic manifolds was not addressed there. As in [17] and [2], we appeal to the theory of h -principle ([10]) for the proof of Theorem 1.1. The result will follow from a general h -principle type result (see Theorem 1.3 stated below) by observing that $Tr_{\alpha}(M, \mathcal{F}_N)$ is the solution space of some open relation which is also invariant under the action of local contactomorphisms.

Theorem 1.3. *Let (M, α) be an open contact manifold and $\mathcal{R} \subset J^r(M, N)$ be an open relation invariant under the action of the pseudogroup of local contact diffeomorphisms of (M, α) . Then the parametric h -principle holds for \mathcal{R} .*

A crucial point about open contact manifolds is that they admit isocontact immersions into arbitrary open neighbourhoods of their cores (see Theorem 3.3). This will follow from a result on equidimensional isocontact immersions which we prove in the paper.

We organise the paper as follows. We recall preliminaries of contact manifolds in Section 2. In Section 3, we prove an equidimensional version of Isocontact Immersion Theorem [4] and obtain Theorem 1.3 as a corollary to it. In Section 4, we briefly review the language of h -principle and a few major results which are necessary for our purpose. We give a proof of Theorem 1.1 and discuss a few corollaries of it in Sections 5 and 6. In the final section we prove Theorem 1.2. For

the sake of completeness we include the relevant background of Γ -structures and its relations to foliations in Section 7.

2. PRELIMINARIES OF CONTACT MANIFOLDS

In this section we review some basic theory of contact manifolds. A hyperplane distribution ξ on a manifold M can be locally written as $\xi = \ker \alpha$ for some local 1-form α on M . The form α is only unique upto multiplication by a nowhere vanishing function. If ξ is coorientable i.e. when $\frac{TM}{\xi}$ is a trivial line bundle, then ξ can be globally written as $\xi = \ker \alpha$ for some 1-form on M [6].

Definition 2.1. Let M be a $2n + 1$ dimensional manifold. A hyperplane distribution ξ is called a *contact structure* if $\alpha \wedge (d\alpha)^n$ is nowhere vanishing for any local 1-form α defining ξ . A global 1-form α for which $\alpha \wedge (d\alpha)^n$ is nowhere vanishing is called a *contact form* on M . The distribution $\ker \alpha$ is then called the contact distribution of α .

Example 2.2. Every odd dimensional Euclidean space \mathbb{R}^{2n+1} has a canonical contact form α given by $\alpha = dz + \sum_{i=1}^n x_i dy_i - y_i dx_i$, where $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ is the canonical coordinate system on \mathbb{R}^{2n+1} .

Every even dimensional Euclidean space \mathbb{R}^{2n} on the other hand has the Liouville form $\lambda = \sum_{i=1}^n (x_i dy_i - y_i dx_i)$, where $(x_1, \dots, x_n, y_1, \dots, y_n)$ is the canonical coordinate system on \mathbb{R}^{2n} . The restriction of λ on the unit sphere in \mathbb{R}^{2n} defines a contact form.

Any contact structure ξ is associated with a conformal symplectic structure defined by the restriction of the 2-form $d\alpha$ on ξ , where α is a local 1-form defining ξ . If α is a contact form then there is a global vector field R_α on M defined by the relations

$$(1) \quad \alpha(R_\alpha) = 1, \quad i_{R_\alpha} d\alpha = 0,$$

where i_X denotes the interior multiplication by the vector field X . Thus, TM has the following decomposition:

$$(2) \quad TM = \ker \alpha \oplus \ker(d\alpha),$$

where $\ker \alpha$ is a symplectic vector bundle with the symplectic form $d'\alpha = d\alpha|_{\ker \alpha}$ and $\ker d\alpha$ is the 1-dimensional subbundle generated by R_α . The vector field R_α is called the *Reeb vector field* of the contact form α .

A contact form α also defines a canonical isomorphism $\phi : TM \rightarrow T^*M$ between the tangent and the cotangent bundles of M given by

$$(3) \quad \phi(X) = i_X d\alpha + \alpha(X)\alpha, \quad \text{where } X \in TM.$$

It is easy to see that the Reeb vector field R_α corresponds to the 1-form α under ϕ .

A vector field X satisfying $\mathcal{L}_X \alpha = f\alpha$ for some smooth function f on M is called a *contact vector field*. For every smooth function H on M , there is a contact vector field $X_H = X_0 + \bar{X}_H$ defined as follows:

$$(4) \quad X_0 = HR_\alpha \quad \text{and} \quad \bar{X}_H \in \Gamma(\xi) \text{ such that } i_{\bar{X}_H} d\alpha|_\xi = -dH|_\xi.$$

Equivalently, X_H is given by

$$(5) \quad \alpha(X_H) = H \quad \text{and} \quad i_{X_H} d\alpha = -dH + dH(R_\alpha)\alpha.$$

The vector field X_H is called the *contact Hamiltonian vector field* of H .

If ϕ_t is a local flow of the contact vector field X , then

$$\frac{d}{dt}\phi_t^*\alpha = \phi_t^*(i_X.d\alpha + d(\alpha(X))) = \phi_t^*(f\alpha) = (f \circ \phi_t)\phi_t^*\alpha.$$

Therefore, $\phi_t^*\alpha = \lambda_t\alpha$, where $\lambda_t = e^{\int f \circ \phi_t dt}$ is a nowhere vanishing function on M . Thus the flow of a contact Hamiltonian vector field preserves the contact structure.

A contact form α defines a bivector field Λ on M given by the relation

$$\Lambda(\beta, \beta') = d\alpha(\phi^{-1}(\beta), \phi^{-1}(\beta')),$$

where β, β' are 1-forms on M . Let $\Lambda^\#(\beta)$ denote the contraction of Λ with a 1-form β on M . The image of the vector bundle morphism $\Lambda^\# : T^*M \rightarrow TM$ is ξ and $\ker \Lambda^\#$ is spanned by $\mathbb{R}\alpha$. The contact Hamiltonian vector field X_H can then be expressed as $X_H = HR_\alpha + \Lambda^\#(dH)$. It may be worth mentioning that the pair (Λ, R_α) defines a Jacobi structure on M .

Definition 2.3. A diffeomorphism $f : (M, \xi) \rightarrow (N, \xi')$ between two contact manifolds with contact structures ξ and ξ' is said to be a *contactomorphism* if $df(\xi) = \xi'$. If the contact structures are defined by global 1-forms α and α' respectively then $f^*\alpha' = \lambda\alpha$ for some nowhere vanishing function λ on M .

Theorem 2.4. Every contact form α on a manifold M of dimension $2n + 1$ can be locally represented as $dz - \sum_{i=1}^n p_i dq_i$, where $(z, q_1, \dots, q_n, p_1, \dots, p_n)$ is a local coordinate system on M .

Let N be a codimension 1 submanifold in a contact manifold (M, α) . If the tangent planes of N are transverse to the contact distribution then there is a codimension 1 distribution D on N obtained as the intersection of $\ker \alpha|_N$ and TN . Since $D = \ker \alpha|_N \cap TN$ is an odd dimensional distribution, $d\alpha|_D$ has a 1-dimensional kernel. If N is locally defined by a function Φ then $d\Phi_x$ does not vanish identically on $\ker \alpha_x$, for $\ker d\Phi_x$ is transversal to $\ker \alpha_x$. Thus there is a unique non-zero vector Y_x in $\ker \alpha_x$ satisfying the relation $i_{Y_x}d\alpha_x = d\Phi_x$. The vector Y_x is called a *characteristic vector* of N at x . Clearly, Y_x is tangent to N at x and it is defined uniquely only up to multiplication by a non-zero real number as Φ is not uniquely determined. However, the 1-dimensional distribution on N defined by Y is uniquely defined by the contact form α . The integral curves of Y are called *characteristics* of N ([1]).

Proposition 2.5. Let M be a contact manifold with contact form α . Suppose that H is a smooth real-valued function on $M \times (-\varepsilon, \varepsilon)$ with compact support such that its graph Γ in $M \times \mathbb{R}^2$ is transversal to the kernel of $\tilde{\alpha} = \alpha - y dt$. Then there is a diffeomorphism $\Psi : M \times (-\varepsilon, \varepsilon) \rightarrow \Gamma$ which pulls back $\tilde{\alpha}|_\Gamma$ onto a multiple of $\alpha \oplus 0$.

Remark: For this theorem to be true it is enough to have the support of H to be contained in $U \times \mathbb{R}$ for some relatively compact open set U of M .

Proof. Define a function $\psi : M \times (-\varepsilon, \varepsilon) \rightarrow M \times \mathbb{R}^2$ by $\psi(x, t) = (x, t, H(x, t))$. Since the image of ψ is Γ , $d\psi$ is transverse to the distribution $\tilde{\xi} = \ker \tilde{\alpha}$ by the hypothesis of the proposition. The map ψ defines a diffeomorphism of $M \times (-\varepsilon, \varepsilon)$ with Γ which pulls back the form $\tilde{\alpha}|_\Gamma$ onto $\alpha - H dt$. It is therefore enough to obtain a diffeomorphism $\Psi : (M \times (-\varepsilon, \varepsilon), \alpha \oplus 0) \rightarrow (M \times (-\varepsilon, \varepsilon), \alpha - H dt)$ which pulls back the 1-form $\alpha - H dt$ onto $\alpha \oplus 0$.

Observe that Γ is globally defined as the zero set of the function $\Phi : M \times \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $\Phi(x, t, s) = H(x, t) - s$. Hence the relation $i_{\bar{X}} d\tilde{\alpha} = d\Phi$ on $\ker \tilde{\alpha}$ uniquely defines the characteristic vector field on Γ . For a fixed t , let X_{H^t} denote the contact Hamiltonian vector field on M for the function H^t on M given by $H^t(x) = H(x, t)$ for all $x \in M$, and consider the vector field \bar{X} on $M \times \mathbb{R}$ as follows: $\bar{X}(x, t) = (X_{H^t}(x), 1)$.

Let $\{\bar{\phi}_s\}$ denote the local flow of \bar{X} on $M \times \mathbb{R}$. Then $\bar{\phi}_s(x, t) = (\phi_s(x, t), s + t)$ for all $x \in M$ and $s, t \in \mathbb{R}$. We define a level preserving map $\Psi : M \times (-\varepsilon, \varepsilon) \rightarrow M \times (-\varepsilon, \varepsilon)$ by

$$\Psi(x, t) = \bar{\phi}_t(x, 0) = (\phi_t(x, 0), t).$$

Since support of H is contained in $K \times (-\varepsilon, \varepsilon)$ for some compact set K , the flow $\bar{\phi}_s$ starting at $(x, 0)$ remains within $M \times (-\varepsilon, \varepsilon)$ for $s \in (-\varepsilon, \varepsilon)$. Note that

$$d\Psi\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t} \bar{\phi}_t(x, 0) = \bar{X}(\bar{\phi}_t(x, 0)) = (X_{H^t}(\phi_t(x, 0)), 1).$$

This implies that

$$\begin{aligned} \Psi^*(\alpha \oplus 0)\left(\frac{\partial}{\partial t}\Big|_{(x,t)}\right) &= (\alpha \oplus 0)(d\Psi\left(\frac{\partial}{\partial t}\Big|_{(x,t)}\right)) = (\alpha \oplus 0)(\bar{X}(\bar{\phi}_t(x, 0))) \\ &= \alpha(\bar{X}(\phi_t(x, 0), t)) = \alpha(X_{H^t}(\phi_t(x, 0))) \\ &= H^t(\phi_t(x, 0)) = H(\bar{\phi}_t(x, 0)) = H(\Psi(x, t)). \end{aligned}$$

Hence, $\Psi^*(\alpha - H dt)\left(\frac{\partial}{\partial t}\right) = 0$. On the other hand, $\Psi^*(\alpha - H dt)|_{M \times \{t\}} = \Psi^*\alpha|_{M \times \{t\}} = \psi_t^*\alpha$, where $\psi_t(x) = \phi_t(x, 0)$, $\psi_0(x) = x$. Now, using the relation

$$\frac{d\phi_t}{dt}(u, y) = X_{t+y}(\phi_t(u, y)),$$

where X_t stands for the vector field X_{H^t} , we get

$$\begin{aligned} \frac{d}{dt} \psi_t^* \alpha &= \psi_t^*(i_{X_t} d\alpha + d(i_{X_t} \alpha)) \\ &= \psi_t^*(dH^t(R_\alpha) \alpha - dH^t + dH^t) \\ &= \psi_t^*(dH^t(R) \alpha) \\ &= \theta(t) \psi_t^* \alpha, \quad \text{where } \theta(t) = \psi_t^*(dH^t(R)). \end{aligned}$$

Hence $\psi_t^* \alpha = e^{\int_0^t \theta(s) ds} \psi_0^* \alpha = e^{\int_0^t \theta(s) ds} \alpha$. Consequently, $\Psi^*(\alpha - H dt) = e^{\int_0^t \theta(s) ds} \alpha$. \square

Remark 2.6. It can be checked that the diffeomorphism Ψ maps the lines in $M \times \mathbb{R}$ onto the characteristics on Γ .

We end this section with the following definitions.

Definition 2.7. Let (N, ξ) be a contact manifold. A monomorphism $F : TM \rightarrow (TN, \xi)$ is called *contact* if F is transversal to ξ and $F^{-1}(\xi)$ is a contact structure on M . A smooth map $f : M \rightarrow (N, \xi)$ is called *contact* if its differential df is contact. Such a map is necessarily an immersion.

Let M be also a contact manifold with a contact structure ξ_0 . A monomorphism $F : TM \rightarrow TN$ is said to be *isocontact* if $\xi_0 = F^{-1}\xi$ and $F : \xi_0 \rightarrow \xi$ is conformal symplectic with respect to the conformal symplectic structures ξ_0 and ξ . A smooth map $f : M \rightarrow N$ is said to be *isocontact* if df is isocontact.

If $\xi = \ker \alpha$ for a globally defined 1-form on N , then f is contact if $f^*\alpha$ is a contact form on M . If $\xi = \ker \alpha$ and $\xi_0 = \ker \alpha_0$ then f is isocontact if $f^*\alpha = \varphi \alpha_0$ for some non-vanishing function $\varphi : M \rightarrow \mathbb{R}$.

Definition 2.8. A submanifold N of a contact manifold (M, ξ) is said to be a *contact submanifold* if the inclusion map $i : N \rightarrow M$ is a contact map. This means that TN is transversal to ξ and the subbundle $TN \cap \xi$ is a contact structure on N . This is equivalent to saying that $TN \cap \xi$ is a symplectic subbundle of $(\xi, d'\alpha)$ [14]. When ξ is defined by a global 1-form α , a submanifold N is contact if and only if $i^*\alpha$ is a contact form on N .

3. EQUIDIMENSIONAL CONTACT IMMERSIONS

Let V and W be two equidimensional manifolds, where V is a manifold with boundary and W is without boundary. Let ξ_t be a homotopy of contact structures on V and $\tilde{\xi}$ a contact structure on W . Let $f_0 : V \rightarrow W$ be an immersion which pulls back $\tilde{\xi}$ onto ξ_0 . We shall prove that there is a regular homotopy f_t of f_0 such that $f_t^*\tilde{\xi} = \xi_t$ for all t .

We first recall a result from [4].

Lemma 3.1. *Let $\alpha_t, t \in [0, 1]$, be a family of contact forms on V and let $\xi_t = \ker \alpha_t$ be the associated contact structures. Then there exists a sequence of primitive 1-forms $\beta_i = r_i ds_i, i = 1, \dots, L$ such that*

- (1) $\alpha_1 = \alpha_0 + \sum_1^L \beta_j$,
- (2) for each $k = 0, \dots, L$ the form $\alpha^{(k)} = \alpha_0 + \sum_1^k \beta_j$ is contact,
- (3) for each $j = 1, \dots, L$ the functions r_j and s_j are supported in a domain which is homeomorphic to a ball.

Theorem 3.2. *Let $\xi_t, t \in [0, 1]$ be a family of contact structures on a compact manifold V with boundary defined by the contact forms α_t . Let $(\tilde{V}, \tilde{\xi} = \ker \eta)$ be a contact manifold without boundary and $f_0 : V \rightarrow \tilde{V}$ an immersion such that $f_0^*\tilde{\xi} = \xi_0$. Then there exists a homotopy of immersions $f_t : V \rightarrow \tilde{V}$ such that $f_t^*(\tilde{\xi}) = \xi_t, t \in [0, 1]$.*

In addition, if V contains a compact submanifold V_0 in its interior and $\xi_t = \xi_0$ on $Op(V_0)$ then f_t can be chosen to be a constant homotopy on $Op(V_0)$.

Proof. We shall show that $f_0 : (V, \xi_0) \rightarrow (\tilde{V}, \tilde{\xi})$ can be homotoped to an immersion $f_1 : V \rightarrow W$ such that $f_1^*\tilde{\xi} = \xi_1$. The stated result is a parametric version of this. In view of Lemma 3.1, it is enough to assume that $\alpha_1 = \alpha_0 + rds$, where r, s are compactly supported and supports are contained in an open set U which is homeomorphic to a ball. Let $\xi_1 = \ker \alpha_1$.

Since r, s are supported on U , the set $\{u \in V : \xi_1 \neq \xi_0 \text{ at } u\}$ is contained in U . Let $f : U \rightarrow U \times \mathbb{R}^2$ be a smooth map defined by

$$f(u) = (u, r(u), s(u)) \text{ for } u \in U.$$

Since r, s are compactly supported there will exist ε_1 and ε_2 such that $Im f$ is contained in $U \times I_{\varepsilon_1} \times I_{\varepsilon_2}$, where I_ε denotes an interval $(-\varepsilon, \varepsilon)$ for any $\varepsilon > 0$. Clearly, $f^*(\alpha \oplus xdy) = \alpha + rds$ and so

$$f : (U, \xi_1) \rightarrow (U \times \mathbb{R}^2, \ker(\alpha_0 \oplus xdy))$$

is an isocontact embedding. Moreover, $f(u) = (u, 0, 0)$ for all $u \in Op(\partial U)$ since the functions r and s are supported within U .

Next we define an isocontact immersion

$$F^0 : (U \times \mathbb{R}^2, \ker(\alpha_0 \oplus xdy)) \rightarrow (\tilde{V} \times \mathbb{R}^2, \ker(\eta \oplus \tilde{x}d\tilde{y}))$$

extending f_0 by $F^0(u, y, x) = (f_0(u), y, x)$. The composition

$$F^0 \circ f : U \xrightarrow{f} U \times \mathbb{R}^2 \xrightarrow{F^0} \tilde{V} \times \mathbb{R}^2$$

is therefore an isocontact immersion which can be extended to an isocontact immersion $F^1 : (V, \xi_1) \rightarrow (\tilde{V} \times \mathbb{R}^2, \ker(\eta \oplus \tilde{x}d\tilde{y}))$ by defining F^1 outside U by $F^1(v) = (f_0(v), 0, 0)$ for $v \in V \setminus U$.

However, we need $\text{Im} F^1$ to fit into \tilde{V} and in order to do so we shall perturb f so that the image goes into $U \times \mathbb{R}$ (compare [8]). The image of the isocontact embedding $f : (U, \xi_1) \rightarrow (U \times I_{\varepsilon_1} \times I_{\varepsilon_2}, \ker(\alpha_0 \oplus xdy))$ is the graph of a smooth function $k = (r, s) : U \rightarrow I_{\varepsilon_1} \times I_{\varepsilon_2}$ which is compactly supported with support contained in the interior of U (as both r and s vanish on a neighbourhood of ∂U). Now let $\pi : U \times I_{\varepsilon_1} \times I_{\varepsilon_2} \rightarrow U \times I_{\varepsilon_1}$ be the projection on the first two factors. Since $\text{Im} f$ is the graph of k , $\pi|_{\text{Im} f}$ is an embedding. Therefore, $\text{Im} f$ is the graph of a smooth function $h : \pi(f(U)) \rightarrow I_{\varepsilon_2}$ which must also be compactly supported. Now extend h to a function $H : U \times I_{\varepsilon_1} \rightarrow I_{\varepsilon_2}$ with compact support in such a way that graph of H is transverse to $\ker(\alpha \oplus xdy)$. Since f is isocontact it is transverse to $\ker(\alpha \oplus xdy)$ and hence graph H is transversal to $\ker(\alpha \oplus xdy)$ on an open neighbourhood of $\pi(f(U))$ for any extension H of h . Then appealing to the genericity of transversal maps we conclude that an arbitrary small perturbation of H has the desired property.]

Let Γ be the graph of H . By Lemma 2.5 there exists a diffeomorphism $\Psi : \Gamma \rightarrow U \times I_{\varepsilon_1}$ with the property that

$$\Psi^*(\ker(\alpha_0 \oplus 0)) = \ker((\alpha_0 \oplus xdy)|_{\Gamma})$$

Since F_0 takes $V \times \mathbb{R}$ into $\tilde{V} \times \mathbb{R}$, the following composition is defined:

$$U \xrightarrow{f} \Gamma \xrightarrow{\Psi} U \times I_{\varepsilon_1} \xrightarrow{F^0} \tilde{V} \times \mathbb{R} \xrightarrow{\pi_{\tilde{V}}} \tilde{V},$$

where $\pi_{\tilde{V}} : \tilde{V} \times \mathbb{R} \rightarrow \tilde{V}$ be the projection onto the second coordinate. Observe that $(\pi_{\tilde{V}})^*\eta = \eta \oplus 0$ and therefore, the composition map $f_1 = \pi_{\tilde{V}} F^0 \Psi f : (U, \xi_1) \rightarrow (\tilde{V}, \tilde{\xi})$ is isocontact. Such a map is necessarily an immersion. This completes the induction argument and hence the proof of the theorem. \square

If M is an open manifold of dimension n then there is a CW complex K of dimension strictly less than n embedded in M such that the inclusion map $i : K \rightarrow M$ is a homotopy equivalence. We shall refer to K as a core of M [4].

Corollary 3.3. *Let (V, ξ) be an open contact manifold and let K be a core of it. Then for a given neighbourhood U of K in V there exists a homotopy of isocontact immersions $f_t : (V, \xi) \rightarrow (V, \xi)$, $t \in [0, 1]$ such that $f_0 = id_V$ and $f_1(V) \subset U$.*

Proof. Since K is a core of V there is an isotopy g_t with $g_0 = id_V$ satisfying the condition $g_1(V) \subset U$. Using a Morse flow on V , we can express V as $V = \bigcup_0^\infty V_i$, where V_0 is a compact neighbourhood of K in U and V_{i+1} is diffeomorphic to $V_i \cup (\partial V_i \times [0, 1])$ so that $\tilde{V}_i \subset \text{Int}(V_{i+1})$ and V_{i+1} deformation retracts onto V_i . If V is a manifold with boundary then this sequence is finite. We shall inductively construct a homotopy of immersions $f_t^i : V \rightarrow V$ with the following properties:

- (1) $f_0^i = id_V$
- (2) $f_1^i(V) \subset U$
- (3) $f_t^i = f_t^{i-1}$ on V_{i-1}

(4) $(f_t^i)^*\xi = \xi$ on V_i .

Assuming the existence of f_t^i , let $\xi_t = (f_t^i)^*(\xi_V)$ and consider a 2-parameter family of contact structures defined by $\eta_{t,s} = \xi_{t(1-s)}$. The parametric version of Theorem 3.2 gives a homotopy of immersions $\tilde{f}_{t,s} : V_{i+2} \rightarrow V$, $(t, s) \in I^2$, satisfying

- (1) $\tilde{f}_{t,0}, \tilde{f}_{0,s} : V_{i+2} \hookrightarrow V$ are the inclusion maps
- (2) $(\tilde{f}_{t,s})^*\xi_t = \eta_{t,s}$; in particular, $(\tilde{f}_{t,1})^*\xi_t = \xi_0 = \xi$
- (3) $\tilde{f}_{t,s} = id$ on V_i since $\eta_{t,s} = \xi_0$ on V_i .

We now extend the homotopy $\{\tilde{f}_{t,s}|_{V_{i+1}}\}$ to all of V as immersions. We denote the extended homotopy by the same notation, by an abuse of notation. We can assume that $\tilde{f}_{0,s} = id_V$ for all s . Define the next level homotopy as follows:

$$f_t^{i+1} = f_t^i \circ \tilde{f}_{t,1} \quad \text{for } t \in [0, 1].$$

This completes the induction step since $(f_t^{i+1})^*(\xi)|_{V_{i+2}} = \xi|_{V_{i+2}}$ and $f_t^{i+1}|_{V_i} = f_t^i|_{V_i}$.

To start the induction we use the isotopy g_t and let $\xi_t = g_t^*\xi$ where ξ is the contact structure associated with the contact form α on V . Note that ξ_t is a family of contact structures on M defined by contact forms. For starting the induction we construct f_t^0 as above by setting $V_{-1} = \emptyset$.

Having constructed the family of homotopies $\{f_t^i\}$ as above we set $f_t = \lim_{i \rightarrow \infty} f_t^i$ which is the desired homotopy of isocontact immersions. □

4. AN h -PRINCIPLE FOR OPEN RELATIONS ON OPEN CONTACT MANIFOLD

We begin with a brief exposition to the theory of h -principle. For further details we refer to [10].

Suppose that M and N be smooth manifolds. Let $J^r(M, N)$ be the space of r -jets of germs of local maps from M to N [9]. There is a canonical map $p^{(r)} : J^r(M, N) \rightarrow M$ which maps a jet $j_f^r(x)$ onto the base point x . A continuous map $\sigma : M \rightarrow J^r(M, N)$ is said to be a *section* of the jet bundle $p^{(r)} : J^r(M, N) \rightarrow M$ if $p^{(r)} \circ \sigma = id_M$. A section of $p^{(r)}$ which is the r -jet of some map $f : M \rightarrow N$ is called a *holonomic* section of the jet bundle.

A subset $\mathcal{R} \subset J^r(M, N)$ of the r -jet space is called a *partial differential relation of order r* (or simply a relation). If \mathcal{R} is an open subset of the jet space then we call it an *open relation*. A C^r map $f : M \rightarrow N$ is said to be a *solution* of \mathcal{R} if the image of its r -jet extension $j_f^r : M \rightarrow J^r(M, N)$ lies in \mathcal{R} .

We denote by $\Gamma(\mathcal{R})$ the space of sections of the bundle $J^r(M, N) \rightarrow M$ having images in \mathcal{R} . The space of C^∞ solutions of \mathcal{R} is denoted by $Sol(\mathcal{R})$. If $Sol(\mathcal{R})$ and $\Gamma(\mathcal{R})$ are endowed with the C^∞ -compact open topology and the C^0 -compact open topology respectively, then the r -jet map

$$j^r : Sol(\mathcal{R}) \rightarrow \Gamma(\mathcal{R})$$

taking a map $f \in Sol(\mathcal{R})$ to j_f^r is continuous. The r -jet map is clearly one to one and its image consists of all holonomic sections of \mathcal{R} .

A differential relation \mathcal{R} is said to satisfy the *h -principle* if every element $\sigma_0 \in \Gamma(\mathcal{R})$ admits a homotopy $\sigma_t \in \Gamma(\mathcal{R})$ such that σ_1 is holonomic. \mathcal{R} satisfies the *parametric h -principle* if the r -jet map $j^r : Sol(\mathcal{R}) \rightarrow \Gamma(\mathcal{R})$ is a weak homotopy equivalence. Thus, the h -principle reduces a differential topological problem to a problem in algebraic topology.

Let $\text{Diff}(M)$ be the pseudogroup of local diffeomorphisms of M . Observe that there is a natural action of $\text{Diff}(M)$ on $J^r(M, N)$ given by $\sigma.\alpha := J_{f \circ \sigma}^r(x)$, where f is a representative of the r -jet α and σ is a local diffeomorphism of M defined near $x \in M$. Let \mathcal{D} be a subgroup of the pseudogroup of local diffeomorphism of N . A differential relation \mathcal{R} is said to be \mathcal{D} -invariant if the following condition is satisfied:

If for some $\alpha \in \mathcal{R}$ and $\sigma \in \mathcal{D}$ the element $\sigma.\alpha$ is defined then it belongs to \mathcal{R} .

The following result, due to Gromov, is the first general result in the theory of h -principle.

Theorem 4.1. *Every open, $\text{Diff}(M)$ invariant relation \mathcal{R} on an open manifold M satisfies the parametric h -principle.*

If the relation is invariant under a smaller group of diffeomorphism \mathcal{D} , then the above theorem will still hold provided \mathcal{D} satisfies some additional properties. We shall see in the next section that the relation associated with the space $Tr_\alpha(M, \mathcal{F}_N)$ is open but not $\text{Diff}(M)$ -invariant. However, it is invariant under the action of local contactomorphisms.

Definition 4.2. A submanifold M_0 of M is said to be sharply movable by a subgroup \mathcal{D} of the pseudogroup $\text{Diff}(M)$, if given any hyper surface S in an open set U in M_0 and for any $\varepsilon > 0$, there is an isotopy δ_t in \mathcal{D} satisfying the following:

- (i) $\delta_0|_U = id_U$,
- (ii) δ_t fixes all points outside the ε -neighbourhood of S ,
- (iii) $\text{dist}(\delta_1(x), M_0) \geq r$ for all $x \in S$ and for some $r > 0$,

where dist denotes the distance with respect to any fixed metric on M .

The following result in h -principle due to Gromov ([10]) plays a crucial role in the proof of Theorem 1.1.

Theorem 4.3. *Let $\mathcal{R} \subset J^r(M, N)$ be an open relation which is invariant under the action of a pseudosubgroup \mathcal{D} of the pseudogroup $\text{Diff}(M)$. If \mathcal{D} sharply moves a submanifold M_0 in M of positive codimension then the parametric h -principle holds for \mathcal{R} on $Op(M_0)$. (Here $Op M_0$ denotes some unspecified open neighbourhood of M_0).*

We now now in a position to prove Theorem 1.3.

Proof. Let \mathcal{D} denote the pseudogroup of contact diffeomorphisms of M . Let M_0 be a submanifold of M of positive codimension. We shall prove that \mathcal{D} sharply moves M_0 at hypersurfaces ([10]). Take a closed hypersurface S in M_0 and an open set $U \subset M$ containing S . Let X be a vector field along S such that X is transversal to M_0 . We obtain a function $H : M \rightarrow \mathbb{R}$ such that

$$\alpha(X) = H, \quad i_X d\alpha|_S = -dH|_S, \quad \text{at points of } S.$$

The contact-Hamiltonian vector field X_H is clearly transverse to M_0 at points of S . As transversality is a stable property and U is small, we can assume that $X_H \pitchfork U$. Now consider the initial value problem

$$\frac{d}{dt}\delta_t(x) = X_H(\delta_t(x)), \quad \delta_0(x) = x$$

The solution to this problem exists for small time t , say for $t \in [0, \varepsilon]$, for all x lying in some small enough neighbourhood of S . Moreover, there would exist a positive real number ε such that the

integral curves $\delta_t(x)$ for $x \in S$ do not meet M_0 during the time interval $(0, \varepsilon)$. Let

$$S_\varepsilon = \cup_{t \in [0, \varepsilon/2]} \delta_t(S).$$

Take a smooth function a which is identically equal to 1 on a small neighbourhood of S_ε and which vanishes outside a slightly bigger neighbourhood of S_ε . We then consider the solution of the same initial value problem with X_H replaced by X_{aH} . Call the new solution $\bar{\delta}_t$. Since X_H is compactly supported the flow of X_H is defined for all time t and it will have the following properties:

- $\bar{\delta}_0|_U = id_U$
- $\bar{\delta}_t = id$ outside a small neighbourhood of S_ε
- $dist(\bar{\delta}_1(x), M_0) > r$ for all $x \in S$ and for some $r > 0$ (because of the choice of ε).

This proves that \mathcal{D} sharply moves any submanifold of M of positive codimension.

Since M is open it has a core K which is of positive codimension. Since the relation \mathcal{R} is invariant under the action of \mathcal{D} , we can apply Theorem 4.3 to conclude that \mathcal{R} satisfies the parametric h -principle near K . We now need to lift the h -principle from OpK to all of M . An arbitrary section F_0 of \mathcal{R} admits a homotopy F_t in $\Gamma(\mathcal{R}|_U)$ such that F_1 is holonomic on U , where U is an open neighbourhood of K in M .

Let $f_t = p^{(r)} \circ F_t$, where $p^{(r)} : J^r(M, N) \rightarrow N$ is the canonical projection map of the jet bundle. By 3.3 above we get a homotopy of isocontact immersions $g_t : (M, \alpha) \rightarrow (M, \alpha)$ satisfying $g_0 = id_M$ and $g_1(M) \subset U$. The concatenation of the homotopies $(F_0 \circ dg_t, f_0 \circ g_t)$ and $(F_t \circ dg_1, f_t \circ g_1)$ gives the desired homotopy in $\Gamma(\mathcal{R})$ between F_0 and the holonomic section $F_1 \circ dg_1$. This proves that $j^r : Sol \mathcal{R} \rightarrow \Gamma(\mathcal{R})$ induces a surjective map at the π_0 level.

On the other hand, if $f_0, f_1 \in Sol(\mathcal{R})$ are such that there exists a path $F_t \in \Gamma(\mathcal{R})$ joining $j^r f_1$ and $j^r f_2$, then there is a homotopy f_t consisting of solutions of \mathcal{R} over some open neighbourhood U of K . The concatenation of the following three paths defines the desired homotopy between f_0 and f_1 :

$$\begin{aligned} M &\xrightarrow{g_t} M \xrightarrow{f_0} N \\ M &\xrightarrow{g_1} M \xrightarrow{f_t} N \\ M &\xrightarrow{g_1-t} M \xrightarrow{f_1} N \end{aligned}$$

where g_t is an isotopy of isocontact immersions of M as described above. Note that all the three paths above are in $Sol(\mathcal{R})$, as \mathcal{R} is invariant under the action of local contactomorphisms. This proves the 1-parametric h -principle for \mathcal{R} . The proof of parametric h -principle is similar. \square

5. PROOF OF THEOREM 1.1

We assume that the reader is familiar with the definition of foliations on a manifold. A foliation \mathcal{F}_N on a manifold N is equivalent to an integrable distribution on N . We shall refer to this distribution as the tangent distribution and denote it by the symbol $T\mathcal{F}_N$. The integral submanifolds of the distribution are called the leaves of the foliation. We denote the quotient vector bundle $TN/T\mathcal{F}_N$ by $\nu(\mathcal{F}_N)$ and call it the normal bundle of the foliation \mathcal{F}_N . Denote the quotient map $TN \rightarrow \nu(\mathcal{F}_N)$ by q .

A smooth map $f : M \rightarrow N$ is said to be transverse to the foliation \mathcal{F}_N if $df_x(T_x M) + (T\mathcal{F}_N)_{f(x)} = T_{f(x)} N$ for all $x \in M$, in other words, $q \circ df : TM \rightarrow \nu(\mathcal{F}_N)$ is an epimorphism. It is known that the inverse image of $T\mathcal{F}_N$ under df is an integrable distribution on M and so

it defines a foliation on M which we denote by $f^*\mathcal{F}_N$. The leaves of $f^*\mathcal{F}_N$ are the preimages of the leaves of \mathcal{F}_N under f . We shall call this foliation *inverse image foliation* of \mathcal{F}_N under f .

Let \mathcal{R} denote the first order differential relation consisting of all 1-jets (x, y, G) , where $x \in M, y \in N$ and $G : T_x M \rightarrow T_y N$ is a linear map such that

- (1) $q \circ G : T_x M \rightarrow \nu(\mathcal{F}_N)_y$ is an epimorphism
- (2) $\ker(q \circ G)$ is transverse to $\ker \alpha_x$ and
- (3) $\ker(q \circ G) \cap \ker \alpha_x$ is a symplectic subspace of $(\ker \alpha_x, d'\alpha_x)$.

The space of sections of \mathcal{R} coincides with $\mathcal{E}_\alpha(TM, \nu(\mathcal{F}_N))$ defined in the introduction.

Proposition 5.1. *Let (M, α) be a contact manifold and (N, \mathcal{F}_N) a foliated manifold. The solution space of \mathcal{R} is the same as $Tr_\alpha(M, \mathcal{F})$.*

Proof. It is sufficient to observe that the following two statements are equivalent (2.8):

- (S1) $f : M \rightarrow N$ is transverse to \mathcal{F}_N and the leaves of the inverse foliation $f^{-1}\mathcal{F}_N$ are contact submanifolds (immersed) of M .
- (S2) $q \circ df$ is an epimorphism and $\ker(q \circ df) \cap \ker \alpha$ is a symplectic subbundle of $(\ker \alpha, d'\alpha)$.

□

We will now show that the relation \mathcal{R} is open and invariant under the action of local contactomorphisms.

Lemma 5.2. *The relation \mathcal{R} defined above is an open relation.*

Proof. Let V be a $(2m+1)$ -dimensional vector space with a (linear) 1-form θ and a 2-form τ on it such that $\theta \wedge \tau^m$ is non-vanishing on V . A $2k+1$ -dimensional subspace K of V will be called an almost contact subspace if $\theta \wedge \tau^k|_K$ is non-zero; equivalently, if K is transversal to $\ker \alpha$ and $K \cap \ker \theta$ is a symplectic subspace of $(\ker \theta, \tau|_{\ker \theta})$. Let W be a vector space of dimension $2n$ and Z is a subspace of W of even codimension. Denote by $L_Z^\natural(V, W)$ the set of all linear maps $L : V \rightarrow W$ which are transverse to Z . This is clearly an open subset in the space of all linear maps from V to W . Define a subset \mathcal{L} of $L_Z^\natural(V, W)$ consisting of all those $L : V \rightarrow W$ for which $\ker L$ is an almost contact subspace of V . We will prove that \mathcal{L} is an open subset of $L_Z^\natural(V, W)$.

Consider the map $E : L_Z^\natural(V, W) \rightarrow Gr_{2(m-n)+1}(V)$ given by $L \mapsto \ker(q \circ L)$, where $q : W \rightarrow W/Z$ is the quotient map. We first show that E is continuous. To see this take $L_0 \in L_Z^\natural(V, W)$ and let $K_0 = \ker(q \circ L_0)$. Consider the subbasic open set \mathcal{U} consisting of all subspaces Y of V such that the canonical projection $p : K_0 \oplus K_0^\perp \rightarrow K_0$ maps Y isomorphically onto K_0 . The inverse image of \mathcal{U} under E consists of all $L : V \rightarrow W$ such that $p|_{\ker(q \circ L)} : \ker(q \circ L) \rightarrow K_0$ is onto. It may be seen easily that if $L \in L_Z^\natural(V, W)$ then

$$\begin{aligned} p \text{ maps } \ker(q \circ L) \text{ onto } K_0 &\Leftrightarrow \ker(q \circ L) \cap K_0^\perp = \{0\} \\ &\Leftrightarrow q \circ L|_{K_0^\perp} : K_0^\perp \rightarrow W/Z \text{ is an isomorphism.} \end{aligned}$$

Now, the set of all L such that $q \circ L|_{K_0^\perp}$ is an isomorphism is an open subset. Hence $E^{-1}(\mathcal{U})$ is open and therefore E is continuous.

Let \mathcal{U}_c denote the subset of $G_{2(m-n)+1}(V)$ consisting of all almost contact subspaces K of V . Then \mathcal{U}_c is open in $G_{2(m-n)+1}(V)$. To see this let $K_0 \in \mathcal{U}$. Recall that a subbasic open set U_K containing K can be identified with the space $L(K, K^\perp)$ of all linear maps from K to K^\perp , where

K^\perp denotes the orthogonal complement of K with respect to some inner product on V ([15]). Let Θ denote the following composition of continuous maps:

$$U_K \cong L(K, K^\perp) \xrightarrow{\Phi} L(K, V) \xrightarrow{\Psi} \Lambda^{2(m-n)+1}(K^*) \cong \mathbb{R}$$

where $\Phi(L) = I + L$ and $\Psi(L) = L^*(\theta \wedge \tau^{2(m-n)+1})$. It is now easy to see that

$$\mathcal{U}_c \cap U_{K_0} = (\Psi \circ \Phi)^{-1}(\mathbb{R} \setminus 0)$$

which proves that $\mathcal{U}_c \cap U_{K_0}$ is open. Since U_K is a subbasic open set in the topology of Grassmannian it proves the openness of \mathcal{U}_c . Combining with the fact that E is continuous, we conclude that \mathcal{L} is open.

We now show that \mathcal{R} is an open relation. Let ξ denote the contact distribution $\ker \alpha$. The restriction of $d\alpha$ to ξ makes it a symplectic vector bundle and therefore, $(\xi, d\alpha|_\xi)$ is locally equivalent to the (trivial) symplectic vector bundle $(U \times \mathbb{R}^{2m}, \omega_0)$, where ω_0 restricts to the canonical symplectic form on each fibre. On the other hand, we can choose a trivializing neighbourhood \tilde{U} for the tangent bundle TN such that $T\mathcal{F}_N$ is isomorphic to $\tilde{U} \times Z$ for some codimension $2q$ -vector space in \mathbb{R}^{2n} . This implies that $\mathcal{R} \cap J^1(U, \tilde{U})$ is diffeomorphic to $U \times \tilde{U} \times \mathcal{L}$. Since the sets $J^1(U, \tilde{U})$ form a subbase for the topology of the jet space, this completes the proof of the lemma. \square

Lemma 5.3. *\mathcal{R} is invariant under the action of the contactomorphism group of (M, α) .*

Proof. Let δ be a local diffeomorphism on an open neighbourhood of $x \in M$ such that $\delta^*\alpha = \lambda\alpha$, where λ is a nowhere vanishing function on $Op\,x$. This implies that $D\delta_x(\xi_x) = \xi_{\delta(x)}$ and $D\delta_x$ preserves the conformal symplectic structure determined by $d\alpha$ on $\ker \xi$. If f is a local solution of \mathcal{R} at $\delta(x)$, then

$$d\delta_x(\ker D(f \circ \delta)_x \cap \xi_x) = \ker Df_{\delta(x)} \cap \xi_{\delta(x)}.$$

Hence $f \circ \delta$ is also a local solution of \mathcal{R} at x . Since \mathcal{R} is open every representative function of a jet in \mathcal{R} is a local solution of \mathcal{R} . Thus local contacto-morphisms act on \mathcal{R} by $\delta.j_f^1(\delta(x)) = j_{f \circ \delta}^1(x)$. \square

Proof. (Proof of 1.1) The theorem follows directly from Theorems 1.3, 5.2 and 5.3. \square

If the foliation on \mathcal{F}_N in Theorem 1.1 is the zero-dimensional foliation then we have the following:

Corollary 5.4. *Let (M, α) be an open contact manifold. The space of smooth submersions $f : M \rightarrow N$ for which the level sets $f^{-1}(x)$, $x \in N$, define contact foliations on M has the same weak homotopy type as the space of epimorphisms $F : TM \rightarrow TN$ for which $\ker F \cap \ker \alpha$ is a symplectic subbundle of $(\ker \alpha, d'\alpha)$.*

Remark 5.5. By the hypothesis of the above corollary, $(\ker F_0, \eta|_{F_0}, d\eta|_{F_0})$ is an almost contact distribution. Since M is an open manifold, by a theorem of A. Phillips (see [18]) the bundle epimorphism $F_0 : TM \rightarrow TN$ can be homotoped (in the space of bundle epimorphism) to the derivative of a submersion f . Hence the distribution $\ker F_0$ is homotopic to an integrable distribution, namely the one given by the submersion f . It then follows from a result in [3] that $(\ker F_0, \eta|_{F_0}, d\eta|_{F_0})$ is homotopic to the distribution associated to a contact foliation \mathcal{F}_1 on M . Theorem 1.1 further implies that it is possible to get a foliation \mathcal{F}_1 which is subordinate to α and is defined by a submersion.

6. CONTACT SUBMERSIONS INTO EUCLIDEAN SPACES

Throughout this section M is a manifold with a contact form α . We denote the contact distribution $\ker \alpha$ by ξ and the restriction of $d\alpha$ to ξ by $d'\alpha$. Hence $(\xi, d'\alpha)$ is a symplectic vector bundle as noted before.

Definition 6.1. A smooth submersion $f : (M, \alpha) \rightarrow N$ from a contact manifold (M, α) to N is called a *contact submersion* if the levels sets of f are contact submanifolds of M . We denote the space of contact submersions by $\mathcal{C}_\alpha(M, N)$.

We shall denote the space $\mathcal{E}_\alpha(TM, \nu\mathcal{F}_N)$ by $\mathcal{E}_\alpha(TM, TN)$ when \mathcal{F}_N is the zero-dimensional foliations on N .

We now restrict ourselves to the case when the target manifold N^{2n} is the Euclidean space \mathbb{R}^{2n} in 1.1. Recall from Section 2 that the tangent bundle TM splits as $\ker \alpha \oplus \ker d\alpha$ since M is a contact manifold. Let $P : TM \rightarrow \ker \alpha$ denote the projection morphism relative to this splitting. If \bar{X}_h denotes the image of the contact Hamiltonian vector field X_h associated to a smooth real-valued function h on M under P .

Lemma 6.2. *Let (M, α) be a contact manifold and $f : M \rightarrow \mathbb{R}^{2n}$ be a submersion with coordinate functions f_1, f_2, \dots, f_{2n} . Then the following statements are equivalent:*

- (C1) *f is a contact submersion.*
- (C2) *The restriction of $d\alpha$ defines a symplectic structure on the bundle spanned by $X_{f_1}, \dots, X_{f_{2n}}$.*
- (C3) *The vector fields $\bar{X}_{f_1}, \dots, \bar{X}_{f_{2n}}$ span a symplectic subbundle of $(\xi, d'\alpha)$.*

Proof. If $f : (M, \alpha) \rightarrow \mathbb{R}^{2n}$ is a contact submersion then the following relation holds pointwise:

$$(6) \quad \ker df \cap \ker \alpha = \langle \bar{X}_{f_1}, \dots, \bar{X}_{f_{2n}} \rangle^{\perp_{d'\alpha}},$$

where f_1, f_2, \dots, f_{2n} are coordinate functions of f and $\perp_{d'\alpha}$ denotes the symplectic complement with respect to $d'\alpha$. Indeed, for any $v \in \ker \alpha$,

$$d'\alpha(\bar{X}_{f_i}, v) = -df_i(v), \quad \text{for all } i = 1, \dots, 2n$$

Therefore, $v \in \ker \alpha \cap \ker df$ if and only if $\beta(\bar{X}_{f_i}, v) = 0$ for all $i = 1, \dots, 2n$, that is $v \in \langle \bar{X}_{f_1}, \dots, \bar{X}_{f_{2n}} \rangle^{\perp_{d'\alpha}}$. Thus the equivalence of (C1) and (C3) is a consequence of equivalence of (S1) and (S2) and equivalence of (C2) and (C3) follows from the relation between X_h and \bar{X}_h . \square

An ordered set of vectors $e_1(x), \dots, e_{2n}(x)$ in ξ_x will be called a *symplectic 2n-frame* in ξ_x if the subspace spanned by these vectors is a symplectic subspace of ξ_x with respect to the symplectic form $d'\alpha_x$. Let $T_{2n}\xi$ be the bundle of symplectic 2n-frames in ξ and $\Gamma(T_{2n}\xi)$ denote the space of sections of $T_{2n}\xi$ with the C^0 -compact open topology.

For any smooth submersion $f : (M, \alpha) \rightarrow \mathbb{R}^{2n}$, define the contact gradient of f by

$$\Xi f(x) = (\bar{X}_{f_1}(x), \dots, \bar{X}_{f_{2n}}(x)),$$

where $f_i, i = 1, 2, \dots, 2n$, are the coordinate functions of f . If f is a contact submersion then $\bar{X}_{f_1}(x), \dots, \bar{X}_{f_{2n}}(x)$ span a symplectic subspace of ξ_x for all $x \in M$, and hence Ξf becomes a section of $T_{2n}\xi$.

Theorem 6.3. *Let (M^{2m+1}, α) be an open contact manifold. Then the contact gradient map $\Xi : \mathcal{C}_\alpha(M, \mathbb{R}^{2n}) \rightarrow \Gamma(T_{2n}\xi)$ is a weak homotopy equivalence.*

Proof. As $T\mathbb{R}^{2n}$ is a trivial vector bundle, the map

$$i_* : \mathcal{E}_\alpha(TM, \mathbb{R}^{2n}) \rightarrow \mathcal{E}_\alpha(TM, T\mathbb{R}^{2n})$$

induced by the inclusion $i : 0 \hookrightarrow \mathbb{R}^{2n}$ is a homotopy equivalence, where \mathbb{R}^{2n} is regarded as the vector bundle over $0 \in \mathbb{R}^{2n}$. The homotopy inverse c is given by the following diagram. For any $F \in \mathcal{E}_\alpha(TM, T\mathbb{R}^{2n})$, $c(F)$ is defined by as $p_2 \circ F$,

$$\begin{array}{ccccc} TM & \xrightarrow{F} & T\mathbb{R}^{2n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n} & \xrightarrow{p_2} & \mathbb{R}^{2n} \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & \mathbb{R}^{2n} & \longrightarrow & 0 \end{array}$$

where p_2 is the projection map onto the second factor.

Since $d'\alpha$ is non-degenerate, the contraction of $d'\alpha$ with a vector $X \in \ker \alpha$ defines an isomorphism

$$\phi : \ker \alpha \rightarrow (\ker \alpha)^*.$$

We define a map $\sigma : \oplus_{i=1}^{2n} T^*M \rightarrow \oplus_{i=1}^{2n} \xi$ by

$$\sigma(G_1, \dots, G_{2n}) = -(\phi^{-1}(\bar{G}_1), \dots, \phi^{-1}(\bar{G}_{2n})),$$

where $\bar{G}_i = G_i|_{\ker \alpha}$. Then noting that $\ker(G_1, \dots, G_{2n}) \cap \ker \alpha = \langle \phi^{-1}(\bar{G}_1), \dots, \phi^{-1}(\bar{G}_{2n}) \rangle^{\perp_{d'\alpha}}$, we get a map $\tilde{\sigma}$ induced by σ as follows:

$$\tilde{\sigma} : \mathcal{E}(TM, \mathbb{R}^{2n}) \longrightarrow \Gamma(M, T_{2n}\xi),$$

Moreover, the contact gradient map Ξ factors as $\Xi = \tilde{\sigma} \circ c \circ d$:

$$(7) \quad \mathcal{C}_\alpha(M, \mathbb{R}^{2n}) \xrightarrow{d} \mathcal{E}_\alpha(TM, T\mathbb{R}^{2n}) \xrightarrow{c} \mathcal{E}_\alpha(TM, \mathbb{R}^{2n}) \xrightarrow{\tilde{\sigma}} \Gamma(T_{2n}\xi).$$

To see this take any $f : M \rightarrow \mathbb{R}^{2n}$. Then, $c(df) = (df_1, \dots, df_{2n})$, and hence

$$\tilde{\sigma}c(df) = (\tilde{\beta}^{-1}(df_1|_\xi), \dots, \tilde{\beta}^{-1}(df_{2n}|_\xi)) = (\bar{X}_{f_1}, \dots, \bar{X}_{f_{2n}}) = \Xi(f)$$

which gives $\tilde{\sigma} \circ c \circ d(f) = \Xi f$.

We claim that $\tilde{\sigma} : \mathcal{E}_\alpha(TM, \mathbb{R}^{2n}) \rightarrow \Gamma(T_{2n}\xi)$ is a homotopy equivalence. To prove this we define a map $\tau : \oplus_{i=1}^{2n} \xi \rightarrow \oplus_{i=1}^{2n} T^*M$ by the formula

$$\tau(X_1, \dots, X_{2n}) = (i_{X_1}d\alpha, \dots, i_{X_{2n}}d\alpha)$$

which induces a map $\tilde{\tau} : \Gamma(T_{2n}\xi) \rightarrow \mathcal{E}(TM, \mathbb{R}^{2n})$. It is easy to verify that $\tilde{\sigma} \circ \tilde{\tau} = id$. On the other hand, $\hat{G} = (\tau \circ \sigma)(G)$ equals G on $\ker \alpha$ for any $G \in \mathcal{E}(TM, \mathbb{R}^{2n})$. Define a homotopy between G and \hat{G} by $G_t = (1-t)G + t\hat{G}$. Then $G_t = G$ on $\ker \alpha$ and hence $\ker G_t \cap \ker \alpha = \ker G \cap \ker \alpha$. This also implies that each G_t is an epimorphism. Consequently the homotopy lies in $\mathcal{E}_\alpha(TM, \mathbb{R}^{2n})$. This shows that $\tilde{\tau} \circ \tilde{\sigma}$ is homotopic to the identity map.

This completes the proof of the theorem since $d : \mathcal{C}(M, \mathbb{R}^{2n}) \rightarrow \mathcal{E}(TM, T\mathbb{R}^{2n})$ is a weak homotopy equivalence (Theorem 1.1) and $c, \tilde{\sigma}$ are homotopy equivalences. \square

Example 6.4. Let \mathbb{S}^{2N-1} denote the $2N-1$ sphere in \mathbb{R}^{2N}

$$\mathbb{S}^{2N-1} = \{(z_1, \dots, z_{2N}) \in \mathbb{R}^{2N} : \sum_1^{2N} |z_i|^2 = 1\}$$

This is a standard example of a contact manifold where the contact form η is induced from the 1-form $\sum_{i=1}^N (x_i dy_i - y_i dx_i)$ on \mathbb{R}^{2N} . For $N > K$, we consider the open manifold $\mathcal{S}_{N,K}$ obtained from \mathbb{S}^{2N-1} by deleting a $(2K-1)$ -sphere:

$$\mathcal{S}_{N,K} = \mathbb{S}^{2N-1} \setminus \mathbb{S}^{2K-1},$$

where

$$\mathbb{S}^{2K-1} = \{(z_1, \dots, z_{2K}, 0, \dots, 0) \in \mathbb{R}^{2N} : \sum_1^{2K} |z_i|^2 = 1\}$$

Then $\mathcal{S}_{N,K}$ is an contact submanifold of \mathbb{S}^{2N-1} . Let ξ denote the contact structure associated to the contact form η on $\mathcal{S}_{N,K}$. Since $\xi \rightarrow \mathcal{S}_{N,K}$ is a symplectic vector bundle, we can choose a complex structure J on ξ such that $d'\eta$ is J -invariant. Thus, (ξ, J) becomes a complex vector bundle of rank $N-1$.

We define a homotopy $F_t : \mathcal{S}_{N,K} \rightarrow \mathcal{S}_{N,K}$, $t \in [0, 1]$, as follows: For $(x, y) \in \mathbb{R}^{2K} \times \mathbb{R}^{2(N-K)} \cap \mathcal{S}_{N,K}$

$$F_t(x, y) = \frac{(1-t)(x, y) + t(0, y/\|y\|)}{\|(1-t)(x, y) + t(0, y/\|y\|)\|}$$

This is well defined since $y \neq 0$. It is easy to see that $F_0 = id$, F_1 maps $\mathbb{S}^{2(N-K)-1}$ into $\mathcal{S}_{N,K}$ and the homotopy fixes $\mathbb{S}^{2(N-K)-1}$ pointwise. Define $r : \mathcal{S}_{N,K} \rightarrow \{0\} \times \mathbb{R}^{2(N-K)} \cap \mathcal{S}_{N,K} \simeq \mathbb{S}^{2(N-K)-1}$ by

$$r(x, y) = (0, y/\|y\|), \quad (x, y) \in \mathbb{R}^{2K} \times \mathbb{R}^{2(N-K)} \cap \mathcal{S}_{N,K}$$

Then F_1 factors as $F_1 = i \circ r$, where i is the inclusion map, and we have the following diagram:

$$\begin{array}{ccccc} r^*(i^*\xi) & \longrightarrow & i^*\xi & \longrightarrow & \xi \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}_{N,K} & \xrightarrow{r} & \mathbb{S}^{2(N-K)-1} & \xrightarrow{i} & \mathcal{S}_{N,K} \end{array}$$

Hence, $\xi = F_0^*\xi \cong F_1^*\xi = r^*(\xi|_{\mathbb{S}^{2(N-K)-1}})$. Now, $\xi|_{\mathbb{S}^{2(N-K)-1}}$ has a decomposition of the following form [13]:

$$\xi \cong \tau^{N-K-1} \oplus \theta^K,$$

where θ^K is a trivial complex vector bundle of rank K and τ^{N-K-1} is a complementary subbundle. Since $\xi \cong r^*(\xi|_{\mathbb{S}^{2(N-K)-1}})$, it follows that ξ must also have a trivial direct summand θ of rank K . Moreover, θ will be a symplectic subbundle of ξ since the complex structure J is compatible with the symplectic structure of ξ . Thus, $\mathcal{S}_{N,K}$ admits a symplectic $2K$ frame spanning θ . Hence it follows from Theorem 1.1 that there is a contact submersion $f : \mathcal{S}_{N,K} \rightarrow \mathbb{R}^{2K}$ for $N > K$.

7. PRELIMINARIES ON Γ -STRUCTURES

In this section we review some definitions and basic facts about the classifying space of Γ -structures for a topological groupoid Γ following [11]. We also recall the connection between foliations on manifolds and Γ_q structures, where Γ_q is the groupoid of germs of local diffeomorphisms of \mathbb{R}^q . For preliminaries of topological groupoid we refer to [16].

Definition 7.1. (Γ -structure) Let Γ be a topological groupoid over a space B . Let X be a topological space with an open covering $\mathcal{U} = \{U_i\}_{i \in I}$. A 1-cocycle on X over \mathcal{U} with values in Γ is a collection of continuous maps

$$\gamma_{ij} : U_i \cap U_j \rightarrow \Gamma$$

such that

$$\gamma_{ik}(x) = \gamma_{ij}(x)\gamma_{jk}(x), \quad \text{for all } x \in U_i \cap U_j \cap U_k$$

Observe that the above condition implies that γ_{ii} has its image in the space of units which can be identified with B . We call two 1-cocycles $(\{U_i\}_{i \in I}, \gamma_{ij})$ and $(\{\tilde{U}_k\}_{k \in K}, \tilde{\gamma}_{kl})$ equivalent if for each $i \in I$ and $k \in K$, there are continuous maps

$$\delta_{ik} : U_i \cap \tilde{U}_k \rightarrow \Gamma$$

such that

(1)

$$\delta_{ik}(x)\tilde{\gamma}_{kl}(x) = \delta_{il} \text{ for } x \in U_i \cap \tilde{U}_k \cap \tilde{U}_l$$

(2)

$$\gamma_{ji}(x)\delta_{ik}(x) = \delta_{ij}(x) \text{ for } x \in U_i \cap U_j \cap \tilde{U}_k$$

An equivalence class of a 1-cocycle is called a Γ -structure.

For a continuous map $F : Y \rightarrow X$ and $\Sigma = (\{U_i\}_{i \in I}, \gamma_{ij})$ a Γ -structure on X , the pullback Γ -structure $F^*\Sigma$ is defined by $\gamma_{ij} \circ F$ with covering $\{F^{-1}U_i\}_{i \in I}$.

Two Γ -structures Σ_0 and Σ_1 on a topological space X are called homotopic if there exists a Γ -structure Σ on $X \times I$, such that $i_0^*\Sigma = \Sigma_0$ and $i_1^*\Sigma = \Sigma_1$, where $i_0 : X = X \times \{0\} \hookrightarrow X \times I$ and $i_1 : X = X \times \{1\} \hookrightarrow X \times I$.

Definition 7.2. Consider a topological groupoid Γ with space of units B , source map \mathbf{s} and target map \mathbf{t} . Now consider the infinite sequences

$$(t_0, x_0, t_1, x_1, \dots)$$

with $t_i \in [0, 1]$ $x_i \in \Gamma$ such that all but finitely many t_i 's are non-zero and $\mathbf{t}(x_i) = \mathbf{t}(x_j) \forall i, j$. Two such sequences

$$(t_0, x_0, t_1, x_1, \dots)$$

and

$$(t'_0, x'_0, t'_1, x'_1, \dots)$$

are called equivalent if $t_i = t'_i$ for all i and $x_i = x'_i$ for all i with $t_i \neq 0$. The space of all equivalence classes is denoted by $E\Gamma$. The topology on $E\Gamma$ is defined to be the weakest such that the following two set maps

$$t_i : E\Gamma \rightarrow [0, 1] \text{ given by } (t_0, x_0, t_1, x_1, \dots) \mapsto t_i$$

$$x_i : t_i^{-1}(0, 1] \rightarrow \Gamma \text{ given by } (t_0, x_0, t_1, x_1, \dots) \mapsto x_i$$

are continuous. Now two elements of $E\Gamma$, $(t_0, x_0, t_1, x_1, \dots)$ and $(t'_0, x'_0, t'_1, x'_1, \dots)$ are equivalent if $t_i = t'_i \forall i$, and there exists $\gamma \in \Gamma$ such that $x_i = \gamma x'_i \forall i$. The space of equivalence classes is called the *classifying space* of Γ which is denoted by $B\Gamma$. Let $P : E\Gamma \rightarrow B\Gamma$ denote the quotient map.

The maps $t_i : E\Gamma \rightarrow [0, 1]$ project down to maps $u_i : B\Gamma \rightarrow [0, 1]$ such that $u_i \circ P = t_i$. $B\Gamma$ has a natural Γ -structure $\Omega = (\{V_i\}_{i \in I}, \gamma_{ij})$, where $V_i = u_i^{-1}(0, 1]$ and $\gamma_{ij} : V_i \cap V_j \rightarrow \Gamma$ is given by

$$(t_0, x_0, t_1, x_1, \dots) \mapsto x_i x_j^{-1}$$

Definition 7.3. (Numerable Γ -structure) Let X be a topological space. An open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X is called *numerable* if it admits a locally finite partition of unity $\{u_i\}_{i \in I}$, such that $u_i^{-1}(0, 1] \subset U_i$. If a Γ -structure can be represented by a 1-cocycle whose covering is numerable then the Γ -structure is called numerable. Two numerable Γ -structures are called *numerbly homotopic* if there exists a homotopy of numerable Γ -structures joining them.

Theorem 7.4. ([12]) *Let Γ be a topological groupoid and Ω be the universal Γ structure on $B\Gamma$. Then*

- (1) Ω is numerable.
- (2) If Σ is a numerable Γ -structure on a topological space X , then there exists a continuous map $f : X \rightarrow B\Gamma$ such that $f^*\Omega = \Sigma$.
- (3) Let $f_0, f_1 : X \rightarrow B\Gamma$ be two continuous functions, then $f_0^*\Omega$ is numerably homotopic to $f_1^*\Omega$ if and only if f_0 is homotopic to f_1 .

For any two topological groupoids Γ_1, Γ_2 and for a groupoid homomorphism $f : \Gamma_1 \rightarrow \Gamma_2$ there exists a continuous map

$$Bf : B\Gamma_1 \rightarrow B\Gamma_2.$$

Let Γ_q be the groupoid of germs of local diffeomorphisms of \mathbb{R}^q . Γ_q is topologised as follows: For a local diffeomorphism $f : U \rightarrow f(U)$, where U is an open set in \mathbb{R}^q , define $U(f)$ as the set of all germs of f at different points of U . The collection of all such $U(f)$ forms a base of some topology. Γ_q becomes a topological groupoid with this topology. The derivative map gives a groupoid homomorphism

$$d : \Gamma_q \rightarrow GL_q(\mathbb{R})$$

which takes the germ of a local diffeomorphism ϕ of \mathbb{R}^q at x onto $d\phi_x$. Thus, to each Γ_q -structure ω on a topological space M there is an associated (isomorphism class of) q -dimensional vector bundle over M with structure group $\text{Im}(d)$. In fact, if ω is defined by the cocycles γ_{ij} then the cocycles $d \circ \gamma_{ij}$ define a vector bundle over M . This vector bundle is denoted by $\nu(\omega)$. In particular, we have a vector bundle $\nu\Omega$ on $B\Gamma_q$ associated with the universal Γ_q -structure Ω on $B\Gamma_q$. We shall denote this vector bundle by $E\Gamma_q$ as well.

Proposition 7.5. *If $f : M \rightarrow B\Gamma_q$ classifies a Γ_q -structure ω on a topological space X , then $Bd \circ f$ classifies the vector bundle $\nu(\omega)$. In particular, $\nu(\Omega_q) \cong Bd^*E(GL_q(\mathbb{R}))$ and hence $\nu(\omega) \cong f^*\nu(\Omega)$.*

A foliation \mathcal{F} on a manifold can also be defined by the following data: An open covering $\{U_i\}$ of M together with submersions $f_i : U_i \rightarrow \mathbb{R}^q$ such that there exists a family of local diffeomorphisms $h_{ij} : f_i(U_i \cap U_j) \rightarrow f_j(U_i \cap U_j)$ satisfying the commutativity relations

$$h_{ij}f_i = f_j \text{ on } U_i \cap U_j \text{ for all } (i, j) \text{ for which } U_i \cap U_j \neq \emptyset$$

and the cocycle conditions

$$h_{jk}h_{ij} = h_{ik} \text{ on } f_i(U_i \cap U_j \cap U_k).$$

These are referred as *Haefliger cocycles*. The cocycles $\{dh_{ij} : U_i \cap U_j \rightarrow GL_q(\mathbb{R})\}$ define the bundle $\nu(\mathcal{F})$.

A foliation \mathcal{F} as above naturally defines a Γ_q structure $\{U_i, g_{ij}\}$ on M , where $g_{ij}(x)$ is the germ of the diffeomorphism h_{ij} at $f_i(x)$ for $x \in U_i \cap U_j$. In particular, $g_{ii}(x)$ is the germ of the identity map of \mathbb{R}^q at $f_i(x)$ and hence g_{ii} takes values in the units of Γ_q . Since the group of units can be identified with \mathbb{R}^q , g_{ii} is nothing but f_i for all i . Thus, one arrives at a Γ_q -structure represented by 1-cocycles (U_i, g_{ij}) such that

$$g_{ii} : U_i \rightarrow \mathbb{R}^q \subset \Gamma_q$$

are submersions for all i . These are referred as Γ_q -foliations in [12]. It is not difficult to see that there is a 1-1 correspondence between foliations and Γ_q -foliations. The functions $\tau_{ij} : U_i \cap U_j \rightarrow GL_q(\mathbb{R})$ defined by $\tau_{ij}(x) = (d \circ g_{ij})_{f_i(x)}$ for $x \in U_i \cap U_j$ define a vector bundle which is isomorphic to the normal bundle $\nu(\mathcal{F})$ of the foliation \mathcal{F} . Hence, the normal bundle $\nu(\mathcal{F})$ is isomorphic to the vector bundle arising from the Γ_q -structure associated to \mathcal{F} .

Consider the Grassmann bundle

$$Gr_{m-q}(TM) \rightarrow M$$

whose fiber over x is $Gr_{m-q}(T_x M)$. We equip the section space $\Gamma(Gr_{m-q}(TM))$ of this bundle with C^∞ -topology. Since every foliation \mathcal{F} on M of codimension q corresponds to a unique (integrable) distribution on M , we can view the space of codimension q foliations on M , denoted by $Fol^q(M)$, as a subspace of the space of C^∞ sections of $Gr_{m-q}(TM)$.

Definition 7.6. Two foliations \mathcal{F}_0 and \mathcal{F}_1 on a manifold M are said to be *integrably homotopic* if there exists a foliation $\tilde{\mathcal{F}}$ on $M \times \mathbb{R}$ transverse to the trivial foliation of $M \times \mathbb{R}$ by leaves $M \times \{t\}$ ($t \in [0, 1]$) such that the induced foliations on $M \times \{0\}$ and $M \times \{1\}$ coincide with \mathcal{F}_0 and \mathcal{F}_1 respectively.

Let $\mathcal{E}(TM, \nu\Omega)$ be the space of all vector bundle epimorphism $F : TM \rightarrow \nu\Omega$. There is a map

$$\pi_0(Fol^{2q}(M)) \xrightarrow{H} \pi_0(\mathcal{E}(TM, \nu\Omega))$$

defined as follows: If \mathcal{F} is a foliation on M and $f : M \rightarrow B\Gamma_q$ is a classifying map of the foliation so that $f^*\Omega \cong \mathcal{F}$, then define $H([\mathcal{F}])$ as the homotopy class of the vector bundle epimorphisms $TM \rightarrow \nu\Omega$ given by the following diagram (see [12])

$$\begin{array}{ccccc} TM & \longrightarrow & \nu\mathcal{F} \cong f^*(\nu\Omega) & \longrightarrow & \nu\Omega \\ & \searrow & \downarrow & & \downarrow \\ & & M & \xrightarrow{f} & B\Gamma_q \end{array}$$

where $TM \rightarrow \nu(\mathcal{F})$ is the quotient map. We shall refer to H as the Haefliger map.

Theorem 7.7. *The map $\pi_0(Fol^q(M)) \xrightarrow{H} \pi_0(\mathcal{E}(TM, \nu\Omega))$ is bijective.*

8. HOMOTOPY CLASSIFICATION OF CONTACT FOLIATIONS

Throughout this section we assume that M is an open contact manifold with a contact form α and we denote the associated contact structure by ξ . Let $Fol_\alpha^{2q}(M)$ denote the space of contact foliations on M of codimension $2q$ subordinate to α . We topologize it as before by identifying it with a subspace of $\Gamma(Gr_{2(m-q)+1}(T))$. Now, let $\mathcal{E}_\alpha(TM, \nu\Omega)$ be the space of all vector bundle epimorphisms $F : TM \rightarrow \nu\Omega$ such that $\ker F$ is transversal to ξ and the restriction of $d\alpha$ to the

subbundle $\ker \alpha \cap \ker F$ is a symplectic structure on it. It is not difficult to see that the image of $Fol_\alpha^{2q}(M)$ under the Haefliger map is contained $\mathcal{E}_\alpha(TM, \nu\Omega)$, since the kernel of $H(\mathcal{F})$ is $T\mathcal{F}$. Let $H_\alpha : Fol_\alpha^{2q}(M) \rightarrow \mathcal{E}_\alpha(TM, \nu\Omega)$ be defined by $H_\alpha(\mathcal{F}) = H(\mathcal{F})$ for all $\mathcal{F} \in Fol_\alpha^{2q}(M)$.

Definition 8.1. Two contact foliations \mathcal{F}_0 and \mathcal{F}_1 on (M, α) are said to be *integrably homotopic relative to α* if there exists a foliation $\tilde{\mathcal{F}}$ on $M \times \mathbb{R}$ transverse to the trivial foliation of $M \times \mathbb{R}$ by leaves $M \times \{t\}$ ($t \in [0, 1]$) such that the following conditions are satisfied:

- (1) the induced foliation on $M \times \{t\}$ for each $t \in [0, 1]$ is a contact foliation subordinate α
- (2) the induced foliations on $M \times \{0\}$ and $M \times \{1\}$ coincide with \mathcal{F}_0 and \mathcal{F}_1 respectively.

Let $\pi_0(Fol_\alpha^{2q}(M))$ denote the space of integrable homotopy classes of contact foliations. Define $H_\alpha([\mathcal{F}])$ as the homotopy class of the vector bundle epimorphisms $H(\mathcal{F}) : TM \rightarrow \nu\Omega$ in $\mathcal{E}_\alpha(TM, \nu\Omega)$. This is well-defined: For if $\tilde{\mathcal{F}}$ be the integral homotopy between \mathcal{F}_0 and \mathcal{F}_1 and if we denote the induced foliation on $M \times \{t\}$ by \mathcal{F}_t then $H_\alpha(\mathcal{F}_t)$ is a homotopy in $\mathcal{E}_\alpha(TM, \nu\Omega)$.

Now let N be any manifold equipped with a foliation \mathcal{F}_N of codimension $2q$. Define a map

$$\pi_0(Tr_\alpha(M, \mathcal{F}_N)) \xrightarrow{P} \pi_0(Fol_\alpha^{2q}(M))$$

by

$$[f] \mapsto [f^* \mathcal{F}_N]$$

which is well defined again by 7.4. So we have the following commutative diagram

• (4)

$$\begin{array}{ccc} \pi_0(Tr_\alpha(M, \mathcal{F}_N)) & \xrightarrow{P} & \pi_0(Fol_\alpha^{2q}(M)) \\ \cong \downarrow \pi_0(\pi \circ d) & & \downarrow H_\alpha \\ \pi_0(\mathcal{E}_\alpha(TM, \nu\mathcal{F}_N)) & \longrightarrow & \pi_0(\mathcal{E}_\alpha(TM, \nu\Omega)) \end{array}$$

where the left vertical arrow is an isomorphism by 1.1. We now recall a theorem from [11],[5].

Theorem 8.2. *Let Σ be a Γ_q -structure on a manifold M . Then there exists a manifold N , a closed embedding*

$$M \xrightarrow{s} N$$

and a Γ_q -foliation \mathcal{F}_N on N such that $s^(\mathcal{F}_N) = \Sigma$ and s is a cofibration.*

Theorem 8.3. *The map $\pi_0(Fol_\alpha^{2q}(M)) \xrightarrow{H_\alpha} \pi_0(\mathcal{E}_\alpha(TM, \nu\Omega))$ is bijective.*

Proof. Let us first prove surjectivity. So take $(\hat{f}, f) \in Epi_\alpha(TM, \nu\Omega)$.

• (5)

$$\begin{array}{ccc} TM & \xrightarrow{\hat{f}} & \nu\Omega \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & B\Gamma_{2q} \end{array}$$

By Theorem 8.2 there exists a manifold N , equipped with a codimension- $2q$ foliation \mathcal{F}_N and a closed embedding $M \xrightarrow{s} N$ such that $s^*\mathcal{F}_N = f^*\Omega$. Let $f' : N \rightarrow B\Gamma_{2q}$ be a map classifying \mathcal{F}_N , i.e. $f'^*\Omega = \mathcal{F}_N$. Hence $(f' \circ s)^*\Omega = f^*\Omega$ and $(f' \circ s)^*\nu(\Omega) = f^*\nu(\Omega)$. Therefore $f' \circ s$ must also be covered by a bundle epimorphism $\hat{g} : TM \rightarrow \nu(\Omega)$ such that $(\hat{g}, f' \circ s) \in \text{Epi}_\alpha(TM, \nu(\Omega))$ and \hat{g} splits as in the following diagram:

• (6)

$$\begin{array}{ccccc} TM & \xrightarrow{\hat{s}} & \nu\mathcal{F}_N & \longrightarrow & \nu\Omega \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{s} & N & \xrightarrow{f'} & B\Gamma_{2q} \end{array}$$

where $\hat{s} : TM \rightarrow s^*(\nu\mathcal{F}_N) \rightarrow \nu\mathcal{F}_N$. It is not difficult to see that (\hat{s}, s) is an element of $\text{Epi}_\alpha(TM, \nu(\mathcal{F}_N))$. Since $f^*\Omega = (f' \circ s)^*\Omega$, by 7.4 there exists a homotopy

$$M \times I \xrightarrow{G} B\Gamma_{2q}$$

starting at $f' \circ s$ and ending at f . As s is a cofibration the following diagram can be solved for some F so that $F(\cdot, 0) = f'$ and $F(s(x), 1) = f(x)$ for all $x \in M$.

• (7)

$$\begin{array}{ccc} M \times \{0\} & \xrightarrow{i_M} & M \times I \\ \downarrow s \times id_0 & \nearrow f' & \downarrow s \times id_I \\ N \times \{0\} & \xrightarrow{i_N} & N \times I \end{array} \quad \begin{array}{c} \nwarrow G \\ B\Gamma_{2q} \\ \swarrow F \end{array}$$

If we set $f'_t(x) = F(x, t)$ for $x \in N$ and $t \in [0, 1]$ then f factors as $f'_1 \circ s$. Since f_t is a homotopy $f'_t{}^*\nu(\Omega) \cong f'^*\nu(\Omega) = \nu(\mathcal{F}_N)$. Thus we get the following homotopy of vector bundle morphism.

• (8)

$$\begin{array}{ccccc} TM & \xrightarrow{\hat{s}} & \nu(\mathcal{F}_N) & \xrightarrow{a_t} & \nu\Omega \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{s} & N & \xrightarrow{f'_t} & B\Gamma_{2q} \end{array}$$

This homotopy starts at the morphism shown in diagram (6) and ends at the morphism shown at diagram (5). Now the left square of diagram (6) represents an element (\hat{s}, s) of $\text{Epi}_\alpha(TM, \nu\mathcal{F}_N)$ whose homotopy class is mapped to $[(\hat{f}, f)]$ by the bottom map of diagram (4). So in diagram (4) $P \circ (\pi_0(\pi \circ d))^{-1}[(\hat{s}, s)]$ is the required preimage of $[(\hat{f}, f)]$ under H_ω . So we have proved the surjectivity.

Now let us prove injectivity. Let $\mathcal{F}_0, \mathcal{F}_1$ be contact foliations on M and let $H_\alpha([\mathcal{F}_0]) = [(\hat{f}_0, f_0)]$ and $H_\alpha([\mathcal{F}_1]) = [(\hat{f}_1, f_1)]$. Let (\hat{f}, f) be a path in $\text{Epi}_\alpha(TM, \nu\Omega)$ joining (\hat{f}_0, f_0) and (\hat{f}_1, f_1) .

• (9)

$$\begin{array}{ccc} TM \times I & \xrightarrow{\hat{f}} & \nu\Omega \\ \downarrow & & \downarrow \\ M \times I & \xrightarrow{f} & B\Gamma_{2q} \end{array}$$

With out loss of generality we can assume that $f_0^*\Omega = \mathcal{F}_0$ and $f_1^*\Omega = \mathcal{F}_1$. So by 8.2 there exists a manifold N equipped with a foliation \mathcal{F}_N and a closed embedding

$$M \times I \xrightarrow{s} N$$

such that $s^*\mathcal{F}_N = f^*\Omega$. As $s_0^*\mathcal{F}_N = f_0^*\Omega = \mathcal{F}_0$ and $s_1^*\mathcal{F}_N = f_1^*\Omega = \mathcal{F}_1$, so $s_0, s_1 \in Tr_\alpha(M, \mathcal{F}_N)$. In view of diagram (4) it is now enough to show that s_0, s_1 are homotopic in $Tr_\alpha(M, \mathcal{F}_N)$. So now we shall show that ds_0 and ds_1 are homotopic in $Epi_\alpha(TM, \nu\mathcal{F}_N)$. A path is given by the following diagram

• (10)

$$\begin{array}{ccc} TM \times I & \longrightarrow & \nu\mathcal{F}_N \\ \downarrow & & \downarrow \\ M \times I & \xrightarrow{s} & N \end{array}$$

which exists because $s^*\mathcal{F}_N = f^*\Omega$. So we have proved injectivity. \square

Let us now try to understand the space $\pi_0(Epi_\alpha(TM, \nu\Omega))$. It is known that the structure group of a contact manifold can be reduced to $1 \times U(n)$, where $U(n)$ is the unitary group.

Theorem 8.4. *Let (M, α) be an open contact manifold and let $\tau : M \rightarrow BU(n)$ be a map classifying the symplectic vector bundle $\xi = \ker \alpha$. Then there is a bijection between the elements of $\pi_0(\mathcal{E}_\alpha(TM, \nu\Omega))$ and the homotopy classes of triples (f, f_0, f_1) , where $f_0 : M \rightarrow BU(q)$, $f_1 : M \rightarrow BU(n - q)$ and $f : M \rightarrow B\Gamma_{2q}$ such that*

- (1) (f_0, f_1) is homotopic to τ in $BU(n)$ and
- (2) $Bd \circ f$ is homotopic to $Bi \circ f_0$ in $BGL(2q)$.

In other words the following diagrams are homotopy commutative:

$$\begin{array}{ccc} & B\Gamma(2q) & \\ & \downarrow Bd & \\ M \xrightarrow{f_0} BU(q) \xrightarrow{Bi} BGL(2q) & \xrightarrow{f} & B\Gamma(2q) \end{array} \quad \begin{array}{ccc} & BU(q) \times BU(n - q) & \\ & \downarrow \oplus & \\ M \xrightarrow{\tau} BU(n) & \xrightarrow{(f_0, f_1)} & BU(q) \times BU(n - q) \end{array}$$

Proof. An element $(F, f) \in \mathcal{E}_\alpha(TM, \nu\Omega)$ defines a (symplectic) splitting of the bundle ξ as

$$\xi \cong (\ker F \cap \xi) \oplus (\ker F \cap \xi)^{d'\alpha}$$

since $\ker F \cap \xi$ is a symplectic subbundle of ξ . Let F' denote the restriction of F to $(\ker F \cap \xi)^{d'\alpha}$. It is easy to see that $(F', f) : (\ker F \cap \xi)^{d'\alpha} \rightarrow \nu(\Omega)$ is a vector bundle map which is fibrewise isomorphism. If $f_0 : M \rightarrow BU(q)$ and $f_1 : M \rightarrow BU(n - q)$ are continuous maps classifying the vector bundles $\ker F \cap \xi$ and $(\ker F \cap \xi)^{d'\alpha}$ respectively, then the classifying map τ of ξ must be

homotopic to $(f_0, f_1) : M \rightarrow BU(q) \times BU(n-q)$ in $BU(n)$ (Recall that the isomorphism classes of Symplectic vector bundles are classified by homotopy classes of continuous maps into BU [13]). Furthermore, note that $(\ker F \cap \xi)^{d'\alpha} \cong f^*(\nu\Omega) = f^*(Bd^*EGL_{2q}(\mathbb{R}))$; therefore, $Bd \circ f$ is homotopic to f_0 in $BGL(2q)$.

Conversely, take a triple (f, f_0, f_1) such that

$$Bd \circ f \sim Bi \circ f_0 \text{ and } (f_0, f_1) \sim \tau.$$

Then ξ has a symplectic splitting given by $f_0^*EU(q) \oplus f_1^*EU(n-q)$. Further, since $Bd \circ f \sim Bi \circ f_0$, we have $f_0^*EU(q) \cong f^*\nu(\Omega)$. Hence there is an epimorphism $F : \xi \xrightarrow{p_2} f_0^*EU(q) \cong f^*\nu(\Omega)$ whose kernel $f_1^*EU(n-q)$ is a symplectic subbundle of ξ . Finally, F can be extended to an element of $\mathcal{E}_\alpha(TM, \nu\Omega)$ by defining its value on R_α equal to zero. \square

Corollary 8.5. *Let B be a symplectic subbundle of ξ with a classifying map $g : M \rightarrow BU(q)$. The integrable homotopy classes of contact foliations on M with their normal bundles isomorphic to B are in one-one correspondence with the homotopy classes of lifts of $Bi \circ g$ in $B\Gamma_{2q}$.*

We end this article with an example to show that a contact foliation on a contact manifold need not be transversally symplectic, even if its normal bundle is a symplectic vector bundle.

Example 8.6. Let us consider a closed almost-symplectic manifold V^{2n} which is not symplectic (e.g., we may take V to be \mathbb{S}^6) and let ω_V be a non-degenerate 2-form on V defining the almost symplectic structure. Set $M = V \times \mathbb{R}^3$ and let \mathcal{F} be the foliation on M defined by the fibres of the projection map $\pi : M \rightarrow V$. Thus the leaves are $\{x\} \times \mathbb{R}^3$, $x \in V$. Consider the standard contact form $\alpha = dz + xdy$ on the Euclidean space \mathbb{R}^3 and let $\tilde{\alpha}$ denote the pull-back of α by the projection map $p_2 : M \rightarrow \mathbb{R}^3$. The 2-form $\beta = \omega_V \oplus d\alpha$ on M is of maximum rank and it is easy to see that β restricted to $\ker \tilde{\alpha}$ is non-degenerate. Therefore $(\tilde{\alpha}, \beta)$ is an almost contact structure on M . Moreover, $\tilde{\alpha} \wedge \beta|_{T\mathcal{F}}$ is nowhere vanishing.

We claim that there exists a contact form η on M such that its restrictions to the leaves of \mathcal{F} are contact. Recall that there exists a surjective map

$$(T^*M)^{(1)} \xrightarrow{D} \wedge^1 T^*M \oplus \wedge^2 T^*M$$

such that $D \circ j^1(\alpha) = (\alpha, d\alpha)$ for any 1-form α on M . Let

$$P : \wedge^1 T^*M \oplus \wedge^2 T^*M \rightarrow \wedge^1 T^*\mathcal{F} \oplus \wedge^2 T^*\mathcal{F}$$

be the projection defined by the pull-back of forms and let $A \subset \Gamma(\wedge^1 T^*M \oplus \wedge^2 T^*M)$ be the set of all pairs (η, Ω) such that $\eta \wedge \Omega^{n+1}$ is nowhere vanishing and let $B \subset \Gamma(\wedge^1 T^*\mathcal{F} \oplus \wedge^2 T^*\mathcal{F})$ be the set of all pairs whose restriction on $T\mathcal{F}$ is nowhere vanishing. Now set $\mathcal{R} \subset (T^*M)^{(1)}$ as

$$\mathcal{R} = D^{-1}(A) \cap (p \circ D)^{-1}(B).$$

Since both A and B are open so is \mathcal{R} . Now if we consider the fibration $M \xrightarrow{\pi} V$ then it is easy to see that the diffeotopies of M preserving the fibers of π sharply moves $V \times 0$ and \mathcal{R} is invariant under the action of such diffeotopies. So by 4.3 there exists a contact form η on $Op(V \times 0) = V \times \mathbb{D}_\varepsilon^3$ for some $\varepsilon > 0$, and η restricted to each leaf of the foliation \mathcal{F} is also contact. Now take a diffeomorphism $g : \mathbb{R}^3 \rightarrow \mathbb{D}_\varepsilon^3$. Then $\eta' = (id_V \times g)^*\eta$ is a contact form on M . Further, \mathcal{F} is a contact foliation relative to η' since $id_V \times g$ is foliation preserving.

But \mathcal{F} can not be transversal symplectic because then there would exist a closed 2-form β whose restriction to $\nu\mathcal{F} = \pi^*(TV)$ would be non-degenerate. This would imply that V is a symplectic manifold contradicting our hypothesis.

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